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Maximal cluster sets along arbitrary curves ☆

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Abstract

The existence of a dense linear manifold of holomorphic functions on a Jordan domain having except for zero maximal cluster set along any curve tending to the boundary with nontotal oscillation value set is shown.

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Keywords: Cluster set; Curve tending to the boundary; Jordan domain; Dense linear manifold

1. Introduction and notation

Throughout this paper, we will use the following standard notations: $\mathbb{N}$ is the set of positive integers, $\mathbb{C}$ is the complex plane, $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$ is the open unit disk, $B(a, r)$ ($\bar{B}(a, r)$) is the euclidean open (closed, resp.) ball with center $a \in \mathbb{C}$ and radius $r > 0$. Moreover, if G is a domain (:= connected, nonempty open subset) of $\mathbb{C}$, then $H(G)$ will stand for the space of holomorphic functions on G. It becomes a

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completely metrizable space (hence a Baire space) when it is endowed with the compact open topology (see [11, p. 238–239]). Finally, if A is a subset of $\mathbb{C}$ then $\bar{A}$ denotes its closure in $\mathbb{C}$ while ∂A denotes its boundary in the extended complex plane $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. In particular, $\mathbb{T}$ will stand for the unit circle $\partial\mathbb{D}$.

A well-known interpolation theorem due to Weierstrass (see [14, Chapter 15]) asserts that if a domain $G \subset \mathbb{C}$, a sequence $\{a_n\}_{n=1}^{\infty} \subset G$ with no limit points in G—i.e., tending to the boundary—and a sequence $\{w_n\}_{n=1}^{\infty} \subset \mathbb{C}$ are prescribed, then there exists a function $f \in H(G)$ such that $f(a_n) = w_n$ for all $n \in \mathbb{N}$. In particular, if we choose as $\{w_n\}_{n=1}^{\infty}$ an enumeration of the complex numbers having rational real and imaginary parts then a function $f \in H(G)$ with $\{f(a_n) : n \in \mathbb{N}\}$ dense in $\mathbb{C}$ is obtained. Since a dense set with finitely many points deleted continues to be a dense set, we get a function $f \in H(G)$ with maximal cluster set along the set $\{a_n\}_{n=1}^{\infty}$, in the sense expressed in the following paragraph. Note also that, equivalently, the density of $f(A)$ for some $f \in H(G)$ can be achieved for every fixed nonrelatively compact subset A of G.

Assume that G is a domain in $\mathbb{C}$, that $F : G \to \mathbb{C}$ is a function defined on G and that A is a subset of G. The *cluster set of F along A* is defined as the set

$$C_A(F) = \{w \in \mathbb{C} : \text{there exists a sequence } \{z_n\}_{n=1}^{\infty} \subset A \text{ tending to some point of } \partial G \text{ such that } \lim_{n\to\infty} F(z_n) = w\}.$$

It is clear that $C_A(F)$ is always closed and that if $C_A(F) \neq \emptyset$ then A is not relatively compact in G. The reader is referred to [5,13] for surveys of results about cluster sets. If $t_0 \in \partial G$ then the *cluster set of F along A at t_0* is defined as

$$C_A(F, t_0) = \{w \in \mathbb{C} : \text{there exists a sequence } \{z_n\}_{n=1}^{\infty} \subset A \text{ tending to } t_0 \text{ such that } \lim_{n\to\infty} F(z_n) = w\}.$$

Again, $C_A(F, t_0)$ is always closed. In addition,

$$C_A(F) = \overline{\bigcup_{t \in \partial G} C_A(F, t)}.$$

If $A = G$ then the subscript "A" is often deleted and the expression "along A" is dropped.

It is an interesting problem to obtain holomorphic functions with *maximal* cluster sets, that is, with cluster sets equal to $\mathbb{C}$. In [1] it is shown that the functions $f \in H(G)$ having maximal cluster set at every boundary point form a residual subset (i.e. its complement is of first category) in $H(G)$, while in [2] it is proved that for a prescribed nonrelatively compact subset $A \subset G$ the set $\{f \in H(G): \overline{f(A)} = \mathbb{C}\}$ is residual in $H(G)$, from which it is easy to conclude that for A as before there exists a residual subset of $H(G)$ all of whose functions have maximal cluster set along A. An important special instance is that of a *curve in G tending to the boundary*, i.e., a continuous map $\gamma : [0, 1) \to G$ such that $\lim_{u\to 1^-} \gamma(u) = \omega :=$ the infinity point of the one-point compactification of G or, equivalently, such that for each compact set $K \subset G$ there is $u_0 = u_0(K) \in [0, 1)$ with $\gamma(u) \in G \setminus K$ for all $u > u_0$ (in particular if $G = \mathbb{D}$ then γ tends to the boundary if and only if $\lim_{u\to 1^-} |\gamma(u)| = 1$). By abuse of language we sometimes identify $\gamma = \gamma([0, 1))$. From the above-mentioned result of [2] and from the fact that a countable intersection of residual subsets is again residual (so dense) one can extract that if Γ is a given *countable* family of curves in G tending to the boundary then there is a dense subset $M \subset H(G)$ such that $C_\gamma(f)$ is maximal for all $f \in M$ and all $\gamma \in \Gamma$.

In this paper, we obtain that at least for each Jordan domain there exists a *dense linear manifold* of holomorphic functions having—except for zero—maximal cluster set along any curve tending to the boundary with nontotal oscillation value set. Hence, we can say that the set of functions with such approximation property is large *not only topologically but also algebraically*.

2. The main result

By a Jordan domain we mean a domain in $\mathbb{C}$ whose boundary in $\mathbb{C}_\infty$ is a topological image of the unit circle $\mathbb{T}$. If $G \subset \mathbb{C}$ is a domain and $A \subset G$ is nonrelatively compact then its oscillation value set is the (nonempty) set

$$\operatorname{Osc}(A) = \{t \in \partial G : \text{there exists a sequence } \{z_n\}_{n=1}^{\infty} \subset A \text{ with } \lim_{n\to\infty} z_n = t\}.$$

We are now ready to state our main result.

Theorem 2.1. *Let G be a Jordan domain. Then there is a dense linear manifold $\mathcal{D}$ in $H(G)$ such that for every $f \in \mathcal{D}$, except for zero, and every curve $\gamma \subset G$ tending to the boundary with* $\operatorname{Osc}(\gamma) \neq \partial G$ *we have $C_\gamma(f) = \mathbb{C}$. In particular, $f(\gamma)$ is dense in $\mathbb{C}$ for each pair f, γ as before.*

Proof. By the Osgood–Carathéodory theorem (see [9]) there exists an homeomorphism φ from the $\mathbb{C}_\infty$-closure of G onto $\overline{\mathbb{D}}$ whose restriction on G is a holomorphic isomorphism from G onto $\mathbb{D}$. Then if $\mathcal{D}$ were the dense linear manifold obtained for $H(\mathbb{D})$ then the set $\mathcal{D}_1 := \{f \circ \varphi : f \in \mathcal{D}\}$ would be the desired linear manifold in $H(G)$. This is easy. Suffice it to say that: the (right) composition operator $C_\varphi f = f \circ \varphi$ is linear; for every compact subset $K \subset G$ the image $\varphi(K)$ is a compact subset of $\mathbb{D}$ and for every curve $\gamma \subset G$ tending to the boundary ∂G with $\operatorname{Osc}(\gamma) \neq \partial G$ we have that $\varphi(\gamma) \subset \mathbb{D}$ is also a curve tending to the boundary $\partial\mathbb{D}$ with $\operatorname{Osc}(\varphi(\gamma)) \neq \partial\mathbb{D}$. The remaining details are left to the reader.

Hence, we may suppose that $G = \mathbb{D}$ from now on. Assume that $\{P_n^*\}_{n=1}^{\infty}$ is a countable dense subset of $H(\mathbb{D})$ (for instance, an enumeration of the holomorphic polynomials having coefficients with rational real and imaginary parts). Then we consider a sequence $\{P_n\}_{n=1}^{\infty}$ where each P_n^* occurs infinitely many times. We also fix two sequences $\{r_n\}$, $\{s_n\}$ of positive real numbers satisfying $r_1 < s_1 < r_2 < s_2 < \cdots < r_n < s_n < \cdots$ and $\lim_{n\to\infty} r_n = 1 = \lim_{n\to\infty} s_n$. Let us divide $\mathbb{N}$ into infinitely many strictly increasing sequences $\{p(n,j) : j = 1, 2, \ldots\}$ $(n \in \mathbb{N})$. For fixed $n \in \mathbb{N}$ we consider the set $F_n \subset \mathbb{D}$ given by the disjoint union

$$F_n = \bar{B}\left(0, \frac{n}{n+1}\right) \cup \bigcup_{j=J(n)}^{\infty} K_j,$$

where $J(n) := \min\{j \in \mathbb{N}\colon\ r_j > \frac{n}{n+1}\}$ and each K_j is the spiral compact set

$$K_j = \left\{\left(r_j + \frac{s_j - r_j}{4\pi}\theta\right)\exp(i\theta)\colon\ \theta \in [0, 4\pi]\right\}.$$

Observe that each K_j has connected complement and that the sequence $\{K_j\}_{j=1}^{\infty}$ goes to $\mathbb{T}$. Note also that every F_n is closed in $\mathbb{D}$. By $\mathbb{D}_\infty$ we will denote the one-point compactification of $\mathbb{D}$, whereas ω will stand for its infinity point. A simple glance reveals that $\mathbb{D}_\infty \setminus F_n$ is connected (indeed, $\mathbb{D} \setminus F_n$ is connected and $\mathbb{D} \setminus F_n \subset \mathbb{D}_\infty \setminus F_n \subset$ the closure in $\mathbb{D}_\infty$ of $\mathbb{D} \setminus F_n$) and locally connected at ω (by a similar reason). In addition, F_n satisfies the following property: For every compact subset $K \subset \mathbb{D}$ there exists a neighbourhood V of ω in $\mathbb{D}_\infty$ such that no component of the interior F_n^0 of F_n intersects both K and V; indeed, $F_n^0 = B(0, \frac{n}{n+1})$ and for any K we can choose $V := \{\omega\} \cup \{\frac{n}{n+1} < |z| < 1\}$. Under these three topological conditions the Nersesjan theorem (see [12] or [7]) asserts the existence of a function $f_n \in H(\mathbb{D})$ approaching a given continuous function $g_n : F_n \to \mathbb{C}$ with g_n holomorphic in F_n^0 within a prescribed error function (= continuous positive function on F_n) $\varepsilon(z)$. Let $\{q_j\}_{j=1}^{\infty}$ be any fixed dense sequence in $\mathbb{C}$. If we select $\varepsilon(z) := \frac{1-|z|}{n}$ then we obtain

$$|f_n(z) - g_n(z)| < \frac{1-|z|}{n} \quad (z \in F_n), \tag{1}$$

where $g_n : F_n \to \mathbb{C}$ is the function defined as

$$g_n(z) = \begin{cases} P_n(z) & \text{if } z \in \bar{B}\left(0, \dfrac{n}{n+1}\right), \\ q_j & \text{if } z \in K_{p(n,j)} \text{ and } p(n,j) \geqslant J(n), \\ 0 & \text{if } z \in K_{p(k,j)}\ (k \neq n) \text{ and } p(k,j) \geqslant J(n). \end{cases}$$

Observe that, trivially, g_n is continuous on F_n and holomorphic in F_n^0, so Nersesjan's theorem applies properly.

Let us define $\mathcal{D}$ as the linear span

$$\mathcal{D} = \text{span}\,\{f_n\colon\ n \in \mathbb{N}\}.$$

Of course, $\mathcal{D}$ is a linear submanifold of $H(\mathbb{D})$, and $\mathcal{D}$ is dense because $\{f_n\}_{n=1}^{\infty}$ is. Indeed, from (1) we have that

$$|f_n(z) - P_n(z)| < \frac{1}{n} \quad \text{for all } z \in \bar{B}\left(0, \frac{n}{n+1}\right).$$

Then if we fix a function P_m^* there exists a sequence $n_1 < n_2 < \cdots$ with $P_{n_j} = P_m^*$ for all $j \in \mathbb{N}$. Now if $K \subset \mathbb{D}$ is compact then there is $j_0 \in \mathbb{N}$ such that $K \subset \bar{B}(0, \frac{n_j}{n_j+1})$ for every

$j > j_0$. Therefore

$$|f_{n_j}(z) - P_m^*(z)| < \frac{1}{n_j} \quad \text{for all } z \in K \text{ and all } j > j_0,$$

so $f_{n_j} \to P_m^*$ $(j \to \infty)$ uniformly on compact in $H(\mathbb{D})$. Hence, the closure of $\{f_n\colon n \in \mathbb{N}\}$ in $H(\mathbb{D})$ contains the dense set $\{P_m^*\colon m \in \mathbb{N}\}$, which proves the density of $\{f_n\}_{n=1}^{\infty}$.

It remains to show that for every prescribed curve $\gamma \subset G$ as in the hypothesis and for every function $f \in \mathcal{D}\backslash\{0\}$ we have $C_\gamma(f) = \mathbb{C}$. Note that for such function f there exist $N \in \mathbb{N}$ and complex scalars $\lambda_1, \dots, \lambda_N$ such that $\lambda_N \neq 0$ and $f = \lambda_1 f_1 + \cdots + \lambda_N f_N$. Since $\operatorname{Osc}(\gamma) \neq \mathbb{T}$ and γ should escape towards $\mathbb{T}$, this curve must intersect all spirals K_j except finitely many of them; indeed, if this were not the case then the shape of K_js together with the continuity of γ would force γ to make infinitely many windings around the origin while approaching $\mathbb{T}$, which would contradict the hypothesis $\operatorname{Osc}(\gamma) \neq \mathbb{T}$. Therefore, there exists $j_0 \in \mathbb{N}$ such that $p(k, j_0) \geqslant J(N)$ $(k = 1, \dots, N)$ and $\gamma \cap K_{p(N,j)} \neq \emptyset$ $(j \geqslant j_0)$. Choose points $z_j \in \gamma \cap K_{p(N,j)}$ $(j \geqslant j_0)$. Then by (1) we obtain, for every $j \geqslant j_0$,

$$|f_N(z_j) - q_j| = |f_N(z_j) - g_N(z_j)| < \frac{1 - |z_j|}{N} \leqslant 1 - |z_j| \leqslant 1 - r_j$$

and

$$|f_n(z_j)| = |f_n(z_j) - g_n(z_j)| < \frac{1 - |z_j|}{n} \leqslant 1 - r_j \quad (n = 1, \dots, N-1).$$

Hence we get

$$\begin{aligned} |f(z_j) - \lambda_N q_j| &= |\lambda_1 f_1(z_j) + \cdots + \lambda_N f_N(z_j) - \lambda_N q_j| \\ &\leqslant |\lambda_N| \cdot |f_N(z_j) - q_j| + \sum_{n=1}^{N-1} |\lambda_n f_n(z_j)| \\ &< \left(\sum_{n=1}^{N} |\lambda_n|\right)(1 - r_j) \to 0 \quad (j \to \infty). \end{aligned}$$

But since $\lambda_N \neq 0$ the sequence $\{\lambda_N q_j\colon j \in \mathbb{N}\}$ is dense in $\mathbb{C}$, so for given $\alpha \in \mathbb{C}$ there is a sequence $\{j_1 < j_2 < \cdots\} \subset \mathbb{N}$ with $\lambda_N q_{j_k} \to \alpha$ as $k \to \infty$. Now we can select a sequence $\{k(1) < k(2) < \cdots\} \subset \mathbb{N}$ and a point $t \in \mathbb{T}$ with $w_l \coloneqq z_{j_{k(l)}} \to t$ $(l \to \infty)$. Then $\{w_l\}_{l=1}^{\infty} \subset \gamma$ and $f(w_l) = f(w_l) - \lambda_N q_{j_{k(l)}} + \lambda_N q_{j_{k(l)}} \to \alpha$ $(l \to \infty)$, so $\alpha \in C_\gamma(f)$. In other words, $C_\gamma(f) = \mathbb{C}$, as required. □

In view of Theorem 2.1, two natural questions arise, namely

(a) Is it possible to replace the arbitrary curve γ to an arbitrary *sequence*$\{z_n\}_{n=1}^{\infty}$ tending to the boundary (even with $\operatorname{Osc}(\{z_n\}_{n=1}^{\infty}) \neq \partial G$)? The elementary Proposition 2.2 below answers this question in the negative.
(b) It is clear that a similar result to Theorem 2.1 cannot hold if one desires that f belongs to a subspace of bounded functions. But even without this boundedness restriction the statement may be false. For instance, if f is in the Hardy space H^p

(see below) of the unit disk then Fatou's theorem asserts that the radial limit $\lim_{r\to 1^-} f(re^{i\theta})$ exists and is finite for all $\theta\in A$, where $A = A_f$ is a subset of $[0, 2\pi]$ such that the Lebesgue measure of $[0, 2\pi]\setminus A$ is zero, see [6]. Therefore $C_\gamma(f)$ is a singleton for each radial curve $\gamma = \{re^{i\theta}\colon\ r\in[0,1)\}$ $(\theta\in A)$. Nevertheless, making a link to a motivating result mentioned in Section 1, we could ask whether at least for a prescribed *countable* family of curves in $\mathbb{D}$ tending to $\mathbb{T}$ the assertion of Theorem 2.1 holds in H^p. Theorem 2.5 below will provide this time a positive answer, even without the restriction Osc $(\gamma)\neq\mathbb{T}$.

Proposition 2.2. *If $G\subset\mathbb{C}$ is a bounded domain and $f\in H(G)$ then there are a point $t\in\partial G$, a value $\alpha\in\mathbb{C}$ and a sequence $\{z_n\}_{n=1}^{\infty}\subset G$ tending to t such that $\lim_{n\to\infty} f(z_n) = \alpha$.*

Proof. If f has infinitely many zeros then the result follows from the Analytic Continuation Principle. Suppose now that f has finitely many zeros. Define $g = f/P$, where $P\equiv 1$ if f has no zeros whereas $P(z)\equiv(z-a_1)\cdots(z-a_p)$ if $a_1,\dots,a_p$ are the zeros of f, counting according their multiplicities. Then g is in $H(G)$ and has no zeros. Let us fix a sequence $\{K_n\}_{n=1}^{\infty}$ of compact subsets of G which is exhaustive, in the sense that its union is G and $K_n\subset K_{n+1}^0$ $(n\in\mathbb{N})$. Without loss of generality, we can suppose $K_1^0\neq\emptyset$. Choose any point $a\in K_1^0$, so $a\in K_n^0$ for all n. Since g has no zeros, the Minimum Modulus Principle tells us that the minimum of $|g|$ on K_n is attained at some point $a_n\in\partial K_n$, therefore $|g(a_n)|\leqslant|g(a)|$. Then

$$|f(a_n)| = |P(a_n)|\cdot|g(a_n)|\leqslant M := |g(a)|\cdot\sup_{z\in G}|P(z)| \quad (n\in\mathbb{N}),$$

where M is finite because G is bounded. Summarizing, we have obtained a sequence $\{a_n\}_{n=1}^{\infty}\subset G$ such that $\{f(a_n)\}_{n=1}^{\infty}$ is bounded. But the exhaustivity property of $\{K_n\}_{n=1}^{\infty}$ implies that for a given compact set $K\subset G$ there is $n_0\in\mathbb{N}$ with $K\subset K_{n_0}$, so $\{a_n\colon\ n>n_0\}\cap K=\emptyset$, whence the compactness of $\bar{G}$ leads us up to a point $t\in\partial G$ with $b_n\to t$ $(n\to\infty)$ for some subsequence $\{b_n\}_{n=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$. Finally, the boundedness of $\{f(b_n)\}_{n=1}^{\infty}$ guarantees that $f(z_n)\to\alpha$ $(n\to\infty)$ for some $\alpha\in\mathbb{C}$ and some subsequence $\{z_n\}_{n=1}^{\infty}$ of $\{b_n\}_{n=1}^{\infty}$. $\square$

Recall that a sequence $T_n : X\to Y$ $(n\in\mathbb{N})$ of continuous linear mappings between two topological vector spaces X, Y is called *universal* or *hypercyclic* whenever there exists a vector $x\in X$—called universal for $\{T_n\}_{n=1}^{\infty}$—whose orbit $\{T_n x\colon\ n\in\mathbb{N}\}$ is dense in Y. By $\mathcal{U}(\{T_n\})$ we will denote the set of such universal vectors. If this set is dense in Y then we say that $\{T_n\}_{n=1}^{\infty}$ is *densely universal*. See [8] for an excellent survey (updated till 1998) about concepts, history and results related to this topic. The following auxiliary result can be found in [3, Theorem 3.1].

Lemma 2.3. *Assume that X, Y are metrizable topological vector spaces and that X is Baire and separable. Suppose that, for each $k\in\mathbb{N}$, $T_n^{(k)} : X\to Y$ $(n\in\mathbb{N})$ is a sequence of continuous linear mappings between X and Y. Assume that for every k and every* *sequence* $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ *the sequence* $\{T_{n_j}^{(k)}\}_{j=1}^{\infty}$ *is densely universal. Then there exists a dense linear manifold* $M \subset X$ *such that*

$$M \setminus \{0\} \subset \bigcap_{k \in \mathbb{N}} \mathcal{U}(\{T_n^{(k)}\}).$$

If $0 < p < \infty$ then the Hardy space H^p is the class of functions

$$f \in H(\mathbb{D}) \text{ for which } \|f\|_p := \sup_{0<r<1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty.$$

It becomes a Banach space for $1 \leqslant p < \infty$ when endowed with the norm $\|f\|_p$. In the 1980s Bourdon and Shapiro were able to prove that for $p = 2$ there is a residual subset of functions $f \in H^p$ for which the orbit $\{f \circ \psi^n \colon n \in \mathbb{N}\}$ is dense in H^p, where ψ^n is the nth-iterate of an automorphism ψ of $\mathbb{D}$ without fixed points in $\mathbb{D}$ (see [4,15, Chapter 7], where many results of this kind can be found). Their proof equally works for $1 \leqslant p < \infty$ because it is ultimately based on the facts that except for perhaps one point of $\mathbb{T}$ the sequence $\psi^n(z)$ tends to a constant value $\alpha \in \mathbb{T}$ and that for every $\beta \notin \mathbb{D}$ the collection Z_β of polynomials vanishing at β is dense in H^p, which in turn is a consequence of Beurling's approximation theorem, see [6, p. 113–114]. Now we denote by φ_a $(a \in \mathbb{D})$ the automorphism of $\mathbb{D}$ given by $\varphi_a(z) = \frac{z+a}{1+\bar{a}z}$. Observe that $S_{\varphi_a}^{-1} = S_{\varphi_{-a}}$, where S_φ denotes the composition operator on H^p generated by a holomorphic self-mapping $\varphi : \mathbb{D} \to \mathbb{D}$, that is, $S_\varphi f = f \circ \varphi$ for all $f \in H^p$. It is a straightforward exercise to check that if $\{a_n\}_{n=1}^{\infty} \subset \mathbb{D}$ and $a_n \to \alpha \in \mathbb{T}$ then $\varphi_{a_n}(t) \to \alpha$ $(n \to \infty)$ for every $t \in \mathbb{T} \setminus \{-\alpha\}$. Hence, in this case, we obtain for all $f \in Z_\alpha$ that $((f \circ \varphi_{a_n})(t))_n$ converges to $f(\alpha) = 0$ at almost every point t of $\mathbb{T}$, and the Bounded Convergence Lebesgue Theorem guarantees that $S_{\varphi_{a_n}} f \to 0$ in H^p $(n \to \infty)$ for every $f \in Z_\alpha$. Similarly, $S_{\varphi_{-a_n}} f \to 0$ for every $f \in Z_{-\alpha}$ in the same way. On the other hand, recall that both Z_α and $Z_{-\alpha}$ are dense in H^p. Thus, the Hypercyclicity Criterion (see [8] or [15]) applies. This has been a sketch of the proof of the following extension of Bourdon–Shapiro's result.

Lemma 2.4. *Let be prescribed a number* $p \in [1, \infty)$ *and a sequence* $\{a_n\}_{n=1}^{\infty} \subset \mathbb{D}$ *tending to a boundary point. Then the functions* $f \in H^p$ *for which the orbit* $\{f \circ \varphi_{a_n} \colon n \in \mathbb{N}\}$ *is dense in* H^p *form a residual subset.*

With the help of the latter two lemmas we can conclude this section by proving the following theorem. We remark that since H^p-convergence is stronger than local uniform convergence, the manifold $\mathcal{D}$ obtained below becomes dense also in $H(\mathbb{D})$.

Theorem 2.5. *Suppose that* $p \in [1, \infty)$ *and that* Γ *is a countable collection of curves in* $\mathbb{D}$ *tending to the boundary. Then there is a dense linear manifold* $\mathcal{D}$ *in* H^p *such that* $C_\gamma(f) = \mathbb{C}$ *for every* $f \in \mathcal{D} \setminus \{0\}$ *and every* $\gamma \in \Gamma$.

Proof. Since Γ is countable, we can write $\Gamma = \{\gamma_k\colon\ k\in\mathbb{N}\}$ where each γ_k is a curve in $\mathbb{D}$ tending to $\mathbb{T}$, whence for every k we can pick a sequence $\{a_n^{(k)}\colon\ n\in\mathbb{N}\}\subset\gamma_k$ tending to some point $\alpha_k\in\mathbb{T}$. If $\{n_1<n_2<\cdots\}\subset\mathbb{N}$ then we have also that $a_{n_j}^{(k)}\to\alpha_k$ as $j\to\infty$. Thus, by Lemma 2.4 the functions $f\in H^p$ for which the orbit $\{f\circ\varphi_{a_{n_j}^{(k)}}\colon\ j\in\mathbb{N}\}$ is dense in H^p form a residual (so dense) subset of H^p for every $k\in\mathbb{N}$. In other words, each sequence $\{T_{n_j}^{(k)}\}_{j=1}^{\infty}$ $(k\in\mathbb{N})$ is densely universal, where $T_n^{(k)}$ denotes the composition operator $f\in H^p\mapsto f\circ\varphi_{a_n^{(k)}}\in H^p$. But $X\coloneqq H^p =\colon Y$ is a Baire metrizable separable topological vector space, hence Lemma 2.3 yields the existence of a dense linear manifold $\mathcal{D}\subset H^p$ such that $\mathcal{D}\setminus\{0\}\subset\bigcap_{k\in\mathbb{N}}\mathcal{U}(\{T_n^{(k)}\})$.

Finally, take a function $f\in\mathcal{D}\setminus\{0\}$ and a curve $\gamma=\gamma_k\in\Gamma$. Then $f\in\mathcal{U}(\{T_n^{(k)}\})$, which implies that $\{f\circ\varphi_{a_n^{(k)}}\colon\ n\in\mathbb{N}\}$ is dense in H^p, so in $H(\mathbb{D})$. In particular, the set $\{(f\circ\varphi_{a_n^{(k)}})(0)\colon\ n\in\mathbb{N}\}=\{f(a_n^{(k)})\colon\ n\in\mathbb{N}\}$ is dense in $\{g(0)\colon\ g\in H(\mathbb{D})\}=\mathbb{C}$. But $\{a_n^{(k)}\colon\ n\in\mathbb{N}\}\subset\gamma$ and $a_n^{(k)}\to\alpha_k\in\mathbb{T}$, so $\mathbb{C}\subset C_\gamma(f)$ and we are done. □

3. Final remarks

1. An important special case in the framework of the cluster sets is the radial one. If $G=\mathbb{D}$, $t_0\in\partial\mathbb{D}$ and $A=\{u\,t_0\colon\ u\in[0,1)\}$ then we call *radial cluster set at* t_0 to $C_\varrho(F,t_0)\coloneqq C_A(F)=C_A(F,t_0)$. Tenthoff has recently constructed (see [16, Kapitel 3]) a dense set of functions $f\in H(\mathbb{D})$ satisfying the following property: For every $t_0\in\mathbb{T}$, every compact subset $K\subset\mathbb{D}$ with connected complement and every continuous function $g: K\to\mathbb{C}$ with $g\in H(K^0)$, there exists a sequence of functions $t_n: K\to\{u\,t_0\colon\ u\in[0,1)\}$—not necessarily holomorphic nor continuous—such that $\lim_{n\to\infty} t_n(z)=t_0$ for all $z\in K$ and $f\circ t_n\to g$ uniformly on K. If we choose specially $K=\{0\}$ then it is derived the following particular case of Theorem 2.1: There is a dense set of functions $f\in H(\mathbb{D})$ all of whose radial cluster sets $C_\varrho(f,t_0)$ are maximal.
2. In connection with the last remark the following question arises: Is the set $\{f\in H(\mathbb{D})\colon\ C_\varrho(f,t_0)=\mathbb{C}$ for all $t_0\in\mathbb{T}\}$ residual in $H(\mathbb{D})$? We do not know the answer, but we are able at least to show the next result: The set $\{f\in H(\mathbb{D})\colon\ C_\varrho(f,t_0)=\mathbb{C}$ for all t_0 belonging to some residual set $A=A_f\subset\mathbb{T}\}$ is residual in $H(\mathbb{D})$. Indeed, by [1] the functions $f\in H(\mathbb{D})$ with maximal cluster set $C(f,t_0)$ at any $t_0\in\mathbb{T}$ is residual, and by Collingwood's maximality theorem (see [5, Theorem 4.8]) if $F:\mathbb{D}\to\mathbb{C}$ is continuous, γ is a curve in $\mathbb{D}$ terminating at 1 (in particular, γ can be the radius $[0,1)$) and $\gamma_t\coloneqq t\cdot\gamma$ $(t\in\mathbb{T})$ then $C_{\gamma_t}(F,t)=C(F,t)$ on a residual set (depending on F) of points t on $\mathbb{T}$.
3. Proposition 2.2 showed that at least for a bounded domain $G\subset\mathbb{C}$, there is no function in $H(G)$ with maximal cluster set along any sequence $\{z_n\}_{n=1}^{\infty}\subset G$ tending to the boundary ∂G. However, if we drop the amount of sequences $\{z_n\}_{n=1}^{\infty}$ then it is possible to get a positive result. Given $A\subset\mathbb{C}$, we denote by A' the set of its accumulation points in $\mathbb{C}_\infty$.

Proposition 3.1. *Let A be a nonrelatively compact subset of a domain $G \subset \mathbb{C}$. Then the set*

$$\mathcal{M} := \{f \in H(G)\colon\ C_A(f,t) = \mathbb{C} \quad \text{for all } t \in A' \cap \partial G\}$$

is residual in $H(G)$.

Proof. Let $\{t_k\}_{k=1}^{\infty}$ be a countable dense subset of $A' \cap \partial G$. For each k, we choose a sequence $\{a_n^{(k)}\}_{n=1}^{\infty} \subset A$ with $a_n^{(k)} \to t_k$ $(n \to \infty)$. By [2], it is known that the sets $\{f \in H(G)\colon\ C_{\{a_n^{(k)}\colon\ n \in \mathbb{N}\}}(f, t_k) = \mathbb{C}\}$ $(k \in \mathbb{N})$ are residual, hence by Baire's theorem

$$\mathcal{D} := \bigcap_{k \in \mathbb{N}} \{f \in H(G)\colon\ C_{\{a_n^{(k)}\colon\ n \in \mathbb{N}\}}(f, t_k) = \mathbb{C}\}$$

is residual.

Let $f \in \mathcal{D}$ and $t \in A' \cap \partial G$. If we prove that $C_A(f,t) = \mathbb{C}$, then we would have $f \in \mathcal{M}$. Thus, $\mathcal{D} \subset \mathcal{M}$ and $\mathcal{M}$ would be residual.

Let $\{w_n\}_{n=1}^{\infty}$ be a countable dense subset of $\mathbb{C}$. By induction, we can construct an increasing sequence $\{m_n\}_{n=1}^{\infty} \subset \mathbb{N}$ such that

$$|f(a_{m_n}^{(k)}) - w_n| < \frac{1}{n} \quad (k = 1, \ldots, n;\ n \in \mathbb{N}). \tag{2}$$

Fix a value $w \in \mathbb{C}$. There is an increasing sequence $\{i_n\}_{n=1}^{\infty} \subset \mathbb{N}$ with

$$|w_{i_n} - w| < \frac{1}{n} \quad (n \in \mathbb{N}). \tag{3}$$

The point t is an accumulation point of the set

$$\{a_{m_{i_n}}^{(k)}\colon\ k = 1, \ldots, i_n;\ n \in \mathbb{N}\}$$

and there exist an increasing sequence $\{j(n)\}_{n=1}^{\infty} \subset \mathbb{N}$ and a sequence $\{k_n\}_{n=1}^{\infty}$, $1 \leqslant k_n \leqslant i_{j(n)}$, such that

$$a_{m_{i_{j(n)}}}^{(k_n)} \to t \quad (n \to \infty).$$

Moreover, by (2) and (3),

$$\begin{aligned} |f(a_{m_{i_{j(n)}}}^{(k_n)}) - w| &\leqslant |f(a_{m_{i_{j(n)}}}^{(k_n)}) - w_{i_{j(n)}}| + |w_{i_{j(n)}} - w| \\ &< \frac{1}{i_{j(n)}} + \frac{1}{j(n)} \to 0 \quad (n \to \infty). \end{aligned}$$

Hence $w \in C_A(f,t)$ and $C_A(f,t) = \mathbb{C}$ for any $t \in A' \cap \partial G$. The proof is finished. □

Observe that if in particular we consider as A the union of the sets of points of countably many prescribed sequences $\{a_n^{(k)}\}_{n=1}^{\infty} \subset G$ $(k \in \mathbb{N})$ with $a_n^{(k)} \to t_k$ $(n \to \infty)$, where $\{t_k\}_{k=1}^{\infty}$ is a countable dense subset of ∂G, we obtain (from the proof) a residual set of functions in $H(G)$ with maximal cluster set along $\{a_n^{(k)}\}_{n=1}^{\infty}$ at t_k $(k \in \mathbb{N})$ and with maximal cluster set along $\{a_n^{(k)}\colon\ n, k \in \mathbb{N}\}$ also at the rest of the points $t \in \partial G$. This statement raises the following open problem: *Assume that for each* *point* $t \in \partial G$ *we fix a sequence* $\{a_n^{(t)}\}_{n=1}^{\infty} \subset G$ *with* $a_n^{(t)} \to t \ (n \to \infty)$. *Is there any function* $f \in H(G)$ *with maximal cluster set along* $\{a_n^{(t)}\}_{n=1}^{\infty}$ *for any* $t \in \partial G$?

4. In view of Theorem 2.1, it is natural to wonder whether there is an entire function $F : \mathbb{C} \to \mathbb{C}$ such that $C_\gamma(F)$ is maximal for every curve γ tending to ∞. This is *false*. In fact, every nonconstant entire function F satisfies $\lim_{\substack{z \to \infty \\ z \in \gamma}} F(z) = \infty$ along at least one curve $\gamma \to \infty$, see [10, p. 159–161].

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